Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-free

ATAR course examination 2020

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2020/66015

Section One: Calculator-free **35% (49 Marks)**

Question 1 **(3 marks)**

Evaluate exactly $\int_0^{\pi}\left(4\cos^2 x-\sin x\right)dx$.

Solution
$\int_0^{\pi}\left(4\cos^2 x-\sin x\right)dx = \int_0^{\pi}\left(2\left(1+\cos 2x\right)-\sin x\right)dx$ $= \int_0^{\pi}\left(2+2\cos 2x-\sin x\right)dx \quad \ldots (1)$ $= \left[2x+\sin 2x+\cos x\right]_0^{\pi}$ $= \left(2\pi+0-1\right)-\left(0+0+1\right)$ $= 2\pi-2$
Specific behaviours
✓ uses the cosine double angle identity to obtain integrand (1) ✓ anti-differentiates the integrand correctly ✓ evaluates the definite integral correctly

Question 2 **(5 marks)**

Plane $\prod$ has vector equation $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(a) Determine the normal vector $\underset{\sim}{n}$ for plane $\prod$. (3 marks)

Solution
Normal $\underset{\sim}{n} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0(2)-1(-1) \\ 1(1)-3(2) \\ 3(-1)-0(1) \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$
Specific behaviours
✓ states that the normal vector can be found using a cross product of the plane direction vectors ✓ obtains the correct form for each component of the cross product ✓ determines the cross product correctly (or a multiple of this vector)

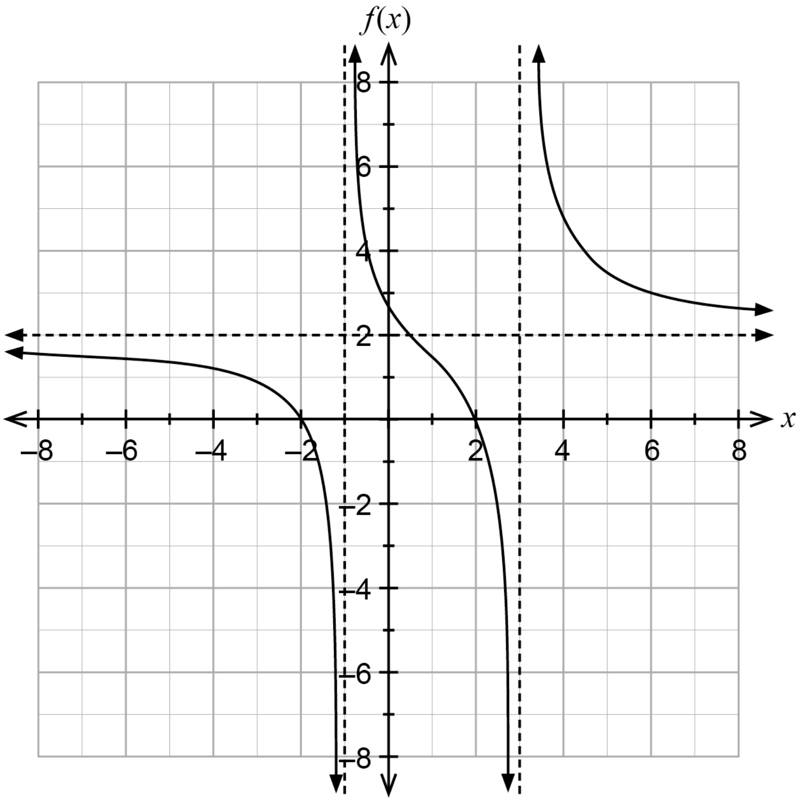
(b) Determine the Cartesian equation for plane $\prod$. (2 marks)

Solution
Equation given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} = -12$ i.e. $x - 5y - 3z = -12$
Specific behaviours
✓ forms the equation using the normal vector $\underset{\sim}{n}$ correctly ✓ determines the Cartesian equation correctly

Question 3 **(6 marks)**

The graph of $y = f(x)$ is shown on the axes below. The defining rule is given by

$f(x) = \dfrac{a(x^2 - b)}{(x+c)(x-d)}$ where a, b, c and d are positive constants.



Determine the value of the constants a, b, c and d. Justify your answers.

a	b	c	d
2	4	1	3

Solution
Horizontal intercepts are $x = -2,\ x = 2\ \ \therefore\ x^2 - b = (x+2)(x-2)$ i.e. $b = 4$ Vertical asymptotes are $x = -1,\ x = 3\ \ \therefore\ c = 1,\ d = 3$ Horizontal asymptote is $y = 2\ \ \therefore\ a = 2$
Specific behaviours
✓ states that $a = 2$ ✓ justifies why $a = 2$ (refers to the horizontal asymptote $y = 2$) ✓ states that $b = 4$ ✓ justifies why $b = 4$ (refers to two horizontal intercepts) ✓ states that $c = 1,\ d = 3$ ✓ justifies why $c = 1,\ d = 3$ (refers to the two vertical asymptotes)

Question 4 **(7 marks)**

Consider the equations for three planes, each written in Cartesian form:

$\Pi_1 \quad x+y+z=4$

$\Pi_2 \quad x-y-z=7$

$\Pi_3 \quad y+z=1$

(a) Explain whether or not any of these planes are parallel. (2 marks)

Solution
$\underset{\sim}{n}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underset{\sim}{n}_2=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ $\underset{\sim}{n}_3=\begin{pmatrix}0\\1\\1\end{pmatrix}$ are the normal vectors for each plane. Since none of the normal vectors are parallel (normal vectors are NOT multiples of each other) then NONE of these planes are parallel.
Specific behaviours
✓ determines the normal vector for each plane ✓ justifies that NONE of the planes are parallel

(b) Solve the given system of simultaneous equations. (3 marks)

Solution
$\Pi_1 \quad x+y+z=4$ $\Pi_2 \quad x-y-z=7$ $\Pi_3 \quad y+z=1$ $(1)+(2): \quad 2x=11 \quad \therefore \quad x=5.5$ $(1)-(3): \quad x=3$ Hence there is NO solution for the system of equations.
Specific behaviours
✓ uses appropriate algebra correctly with TWO pairs of equations ✓ solves correctly to find the first value for x ✓ deduces that there is no solution

Question 4 (continued)

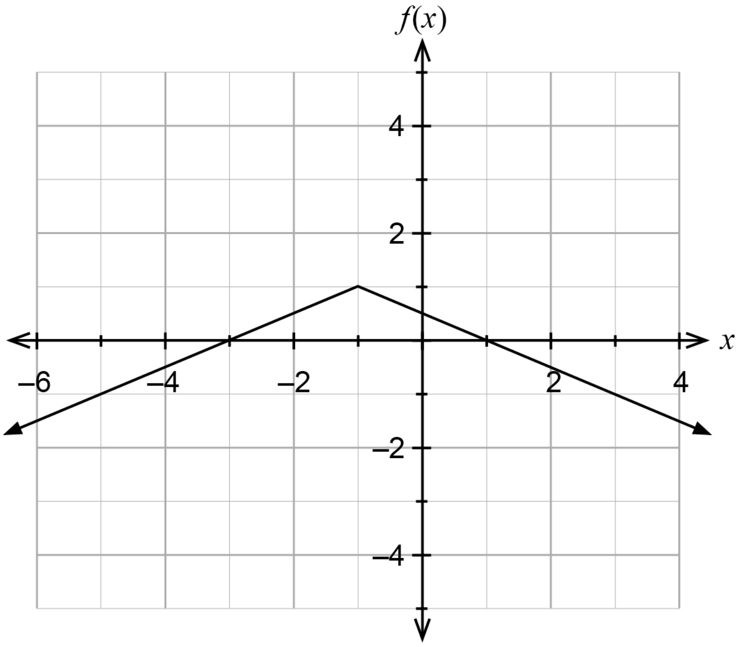
Alternative Solution
$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 7 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_1 + R_2 \\ R_3 \to R_1 - R_3 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 0 & 0 & 11 \\ 1 & 0 & 0 & 3 \end{bmatrix}$ From R_2: $2x = 11$ i.e. $x = 5.5$ From R_3: $x = 3$ Hence there is NO solution for the system of equations.   Note: $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 7 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_2 + R_3} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 7 \\ 1 & 0 & 0 & 8 \end{bmatrix}$ From R_3: $x = 8$
Specific behaviours
✓ applies at least TWO correct row operations   ✓ solves correctly to find the first value for x ✓ deduces that there is no solution

(c) Give the geometric interpretation of the solution for this system of equations. (2 marks)

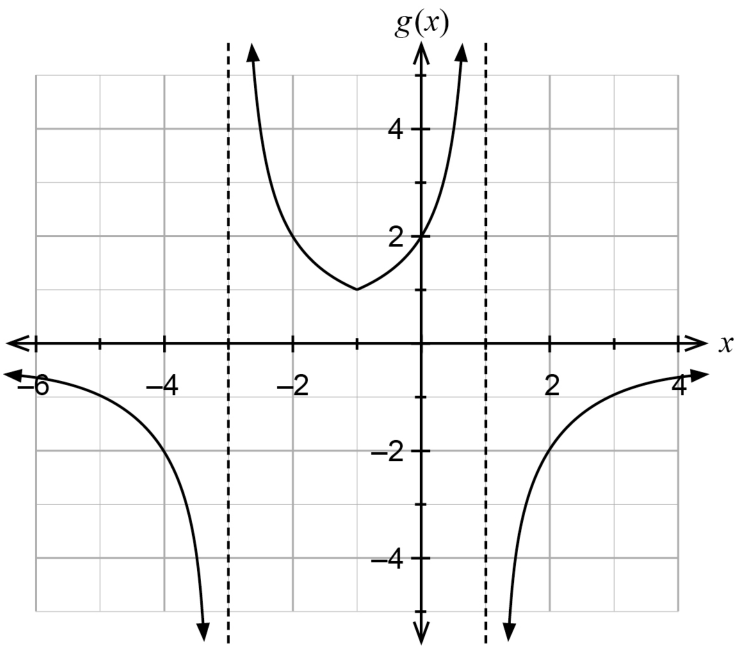
Solution
There is NO intersection between the three planes. We already know that none of the planes are parallel from part (a).   Hence each PAIR of planes intersect in LINES, but that these lines are PARALLEL to each other.   Note: $\Pi_1 \quad x+y+z=4$ $\Pi_2 \quad x-y-z=7$ intersect in line $\underset{\sim}{r} = \begin{pmatrix} 5.5 \\ \lambda \\ 1-\lambda \end{pmatrix}$ i.e. $x = 5.5,\ y+z=1$ $\Pi_1 \quad x+y+z=4$ $\Pi_3 \quad y+z=1$ intersect in line $\underset{\sim}{r} = \begin{pmatrix} 3 \\ \lambda \\ 1-\lambda \end{pmatrix}$ i.e. $x = 3,\ y+z=1$ $\Pi_2 \quad x-y-z=7$ $\Pi_3 \quad y+z=1$ intersect in line $\underset{\sim}{r} = \begin{pmatrix} 6 \\ \lambda \\ 1-\lambda \end{pmatrix}$ i.e. $x = 6,\ y+z=1$ Clearly these lines are PARALLEL since their direction vectors are the same, where $\underset{\sim}{d} = 0\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$. Note that the normal vectors are coplanar since $2\underset{\sim}{n}_3 = \underset{\sim}{n}_1 - \underset{\sim}{n}_2$ i.e. one normal vector is a linear combination of the other two.
✓ states that the three planes have NO intersection   ✓ refers to PAIRS of planes intersecting in LINES that are PARALLEL

Question 5 **(7 marks)**

The graph of $f(x) = 1 - \dfrac{|x+1|}{2}$ is shown below.



(a) Sketch the graph of $g(x) = \dfrac{1}{f(x)}$ on the axes below. (4 marks)



Solution
Shown above.
Specific behaviours
✓ indicates vertical asymptotes at $x=-3$ and $x=1$
✓ indicates $g(x)<0$ as $\lvert x\rvert \to \infty$
✓ indicates $g(x)\geq 1$ for $-3<x<1$
✓ indicates correct graph curvature (cusp at $x=-1$ is not required)

Question 5 (continued)

(b) Hence give the domain and range for $h(x) = \dfrac{4}{2 - |x+1|}$. (3 marks)

Solution
$h(x) = \dfrac{4}{2 - \lvert x+1\rvert} = \dfrac{4}{2\left(1-\dfrac{\lvert x+1\rvert}{2}\right)}$ $= \dfrac{2}{f(x)} = 2g(x)$ Domain $D_h = D_g = \{ x \mid x \neq -3,\ x \neq 1\}$ Range $R_h = \{ y \mid y < 0,\ y \geq 2\}$

Specific behaviours
✓ states the correct domain (from function g)   ✓ states the correct range component $y < 0$ ✓ states the correct range component $y \geq 2$

Question 6 **(13 marks)**

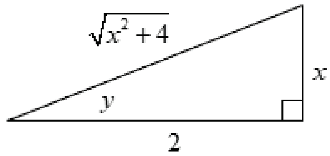
Consider $f(x) = 2\tan(x)$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Let $g(x) = f^{-1}(x)$ be the inverse of function f.

(a) Determine the defining rule for $y = g(x)$. (2 marks)

Solution
Function f: $y = 2\tan(x)$ Hence f^{-1}: $x = 2\tan(y)$ i.e. $\frac{x}{2} = \tan(y)$ $\therefore\ y = \tan^{-1}\left(\frac{x}{2}\right)$ i.e. $g(x) = f^{-1}(x) = \tan^{-1}\left(\frac{x}{2}\right)$ *or* $\arctan\left(\frac{x}{2}\right)$
Specific behaviours
✓ interchanges x, y to form the rule for the inverse function ✓ expresses $f^{-1}(x)$ correctly in terms of x

(b) By using implicit differentiation show that $g'(x)$ can be written in the form $\frac{a}{x^2+b}$.
(4 marks)

Solution
$\frac{d}{dx}(x) = \frac{d}{dx}(2\tan(y))$ $1 = 2(\sec^2 y)\left(\frac{dy}{dx}\right)$ ... (1) $\therefore\ \frac{dy}{dx} = \frac{\cos^2 y}{2} = \frac{1}{2}\left(\frac{2}{\sqrt{x^2+4}}\right)^2 = \frac{2}{x^2+4}$ Given $\tan(y) = \frac{x}{2}$ $\cos(y) = \frac{2}{\sqrt{x^2+4}}$  OR $\frac{dy}{dx} = \frac{1}{2\sec^2 y} = \frac{1}{2(1+\tan^2 y)} = \frac{1}{2+2\left(\frac{x}{2}\right)^2} = \frac{2}{x^2+4}$
Specific behaviours
✓✓ differentiates implicitly correctly to obtain statement (1) ✓ obtains an expression for $\cos(y)$ or $\tan(y)$ correctly in terms of x ✓ obtains a correct simplified expression for $\frac{dy}{dx}$ correctly in the form $\frac{a}{x^2+b}$

Question 6 (continued)

(c) Show that $\dfrac{3x^2+2x+6}{(x^2+4)(x-3)}$ can be expressed as $\dfrac{q}{x^2+4} + \dfrac{r}{x-3}$ and hence determine the values for q and r. (3 marks)

Solution
It is required that $q(x-3) + r(x^2+4) = 3x^2+2x+6$
Hence $rx^2 + qx + (4r-3q) = 3x^2+2x+6$
Equating co-efficients we obtain: $r = 3 \quad ... (1)$
$q = 2 \quad ... (2)$
$4r-3q = 6 \quad ... (3)$
Testing $q=2,\ r=3$ in equation (3) : $4(3)-3(2)=6$ is true.
Solving gives $q=2,\ r=3$.
Specific behaviours
✓ forms the equivalence of numerators correctly ✓ solves for q,r correctly ✓ tests the consistency of q,r to obtain the constant 6

(d) Hence determine $\displaystyle\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx$. (4 marks)

Solution
$\displaystyle\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx = \int \frac{2}{x^2+4} + \frac{3}{x-3}\,dx$
$= \tan^{-1}\left(\dfrac{x}{2}\right) + 3\ln\lvert x-3\rvert + c$
Specific behaviours
✓ re-writes the integrand in terms of the partial fractions correctly ✓ anti-differentiates correctly using the logarithm of an absolute value ✓ uses the result of part (b) to correctly anti-differentiate ✓ uses a constant of integration

Question 7 **(5 marks)**

Evaluate $\int_{-1}^{7} \frac{3x}{\sqrt{x+2}}\,dx$ exactly using the substitution $u=\sqrt{x+2}$.

Solution
When $x=-1,\ u=1$ $\quad \frac{du}{dx} = \frac{1}{2\sqrt{x+2}} \quad \therefore\ dx = 2\sqrt{x+2}\ du$ and $x=7,\ u=3$ $\therefore\ \int_{-1}^{7}\frac{3x}{\sqrt{x+2}}dx = \int_{1}^{3}\frac{3x}{\sqrt{x+2}}\,2\sqrt{x+2}\ du$ $= \int_{1}^{3} 6x\ du$ $= \int_{1}^{3} 6\left(u^2-2\right)du$ $= \int_{1}^{3} 6u^2-12\ du = \left[2u^3-12u\right]_1^3$ $= (54-36)-(2-12)$ $= 28$
Specific behaviours
✓ changes the limits correctly ✓ obtains dx in terms of du correctly ✓ simplifies the integrand correctly ✓ anti-differentiates the integrand correctly ✓ evaluates the definite integral correctly

Question 8 **(3 marks)**

Consider the complex sum: $\sum_{n=1}^{2020} n\,i^n \;=\; 1i^1 + 2i^2 + 3i^3 + ... + 2020i^{2020}$

Express the value of this sum in the form $r\,cis\,\theta$ where $-\pi < \theta \le \pi$.

Solution
$\sum_{n=1}^{4} n\,i^n \;=\; 1(i)^1 + 2(i)^2 + 3(i)^3 + 4(i)^4$ $=\; i \;-\; 2 \;-\; 3i \;+\; 4 \;=\; 2-2i$ $\sum_{n=5}^{8} n\,i^n \;=\; 5(i)^5 + 6(i)^6 + 7(i)^7 + 8(i)^8$ $=\; 5i \;-\; 6 \;-\; 7i \;+\; 8 \;=\; 2-2i$ $\sum_{n=9}^{12} n\,i^n \;=\; 9(i)^9 + 10(i)^{10} + 11(i)^{11} + 12(i)^{12}$ $=\; 9i \;-\; 10 \;-\; 11i \;+\; 12 \;=\; 2-2i$ Hence $\sum_{n=1}^{2020} n\,i^n \;=\; (2-2i)+(2-2i)+(2-2i)+\;...$ $\quad$ 505 terms $=\; 505(2-2i)$ $=\; 1010-1010i$ $=\; 1010\sqrt{2}\;cis\left(-\frac{\pi}{4}\right) \quad or \quad \frac{2020}{\sqrt{2}}cis\left(-\frac{\pi}{4}\right)$
Specific behaviours
✓ evaluates the sum of the first 4 terms correctly   ✓ generalises that the sum of the first 4 terms repeats 505 times   ✓ simplifies correctly in the form $r\,cis\,\theta$

An *Acknowledgements variation* document is available on the Authority website.

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